Beautiful minds

Princeton's mathematicians explore the hidden patterns that underlie nature and form the basis of modern technologies

By Catherine Zandonella

TOWERING ABOVE THE ARCHES AND IVY, 13-story Fine Hall is home to the Department of Mathematics and to some of the deepest thinkers on campus.

"Our goal as a department is to work on the most fundamental problems, to address the most interesting questions, to open new directions of inquiry, and to push the frontiers of mathematics. It is that simple," said David Gabai, the Hughes-Rogers Professor of Mathematics and chair of the department since 2012.

It is a goal that Princeton has met for more than a century. It was here in the late 1930s that Alan Turing — subject of the 2014 film *The Imitation Game* — earned his Ph.D. and worked on ideas that form the foundation of computer science. In the 1990s at Princeton, Andrew Wiles conquered one of math's toughest challenges by proving Fermat's Last Theorem, a problem that had baffled mathematicians for three centuries. It was at Princeton that John Nash, his life also made into a film, *A Beautiful Mind*, did some of his most influential work in mathematics.

Today's mathematics department carries on this tradition of excellence. Graduate students of the program at Princeton have gone on to become professors at the top universities in the country and abroad. "The combination of freedom to explore what interests you and exposure to some of the most renowned faculty in the field makes Princeton's one of the best programs for training new researchers," said Igor Rodnianski, professor of mathematics and acting chair of the department.

The number of Princeton undergraduates choosing math has tripled in the past 15 years, said Rodnianski. "People are starting to understand that mathematics is one of the foundational sciences, and that getting an undergraduate education in math will be helpful no matter what you do afterwards."

Science of patterns

Mathematics is often described as the science of finding patterns, which appear throughout nature, in everything from ripples of pond water to the orbits of the planets. The patterns found in the numbers we use every day (1, 2, 3 …) have fascinated mathematicians since the time of the ancient Greeks. In the last few centuries, mathematics research has led to discoveries about the underlying structure of nature and fueled the development of today's technologies.

"A lot of people think that mathematics is a rigid science, that we just sit around applying existing problem-solving tools — like addition, subtraction, multiplication and division — to problems that are given to us," Rodnianski said. "Well, neither are the problems given to us, nor are we given the tools. We have to develop the tools as we go along."

The creative nature of mathematics research makes its pursuit rewarding in its own right, according to many of the Princeton mathematicians interviewed for this article. Often the applications of this research take decades to emerge, but when they do, they are transformative. "Mathematical concepts can have dramatic consequences in other fields," said Elias Stein, the Albert Baldwin Dod Professor of Mathematics, Emeritus, who is active in research in the field of analysis, which has its roots in the development of calculus. For example, the discovery of the structure of DNA in the early 1950s was possible because of previous work on the mathematics of analyzing signals to identify their components.

Sometimes, curiosity-driven mathematics research finds more immediate use. In the 1990s Peter Sarnak, the Eugene Higgins Professor of Mathematics, and two fellow mathematicians, Alexander Lubotzky of the Hebrew University of Jerusalem and Ralph Phillips of Stanford University, introduced

Ramanujan graphs, which are arrangements of dots, with pairs of dots joined by edges according to specialized rules of number theory. They showed that for these graphs, the dots were highly interconnected, but with the fewest possible connections. This turned out to be of great interest in computer networking, where engineers could cut costs by using fewer wires between nodes without sacrificing connectivity. Other examples of curiosity-driven investigation leading to practical applications include the research of Professor of Mathematics Assaf Naor on the field of computer science and the research of Charles Fefferman, the Herbert E. Jones, Jr. '43 University Professor of Mathematics, on the study of fluids.

New findings and insights in mathematics can take years to develop, but the effort pays off, said Gabai, who is on sabbatical at the nearby Institute for Advanced Study, a leading center for theoretical research. "A career in mathematics research is about enduring frustrations with tenacity, being willing to be stuck on a problem, having an open mind and trying things others haven't," Gabai said. "The path is not easy, but it is incredibly rewarding in the end."

Einstein's legacy

Although Albert Einstein was never on the faculty at Princeton, he occupied an office in the University's mathematics building in the 1930s while waiting for construction of the Institute for Advanced Study, and his ideas have inspired generations of physicists and mathematicians at Princeton and around the world.

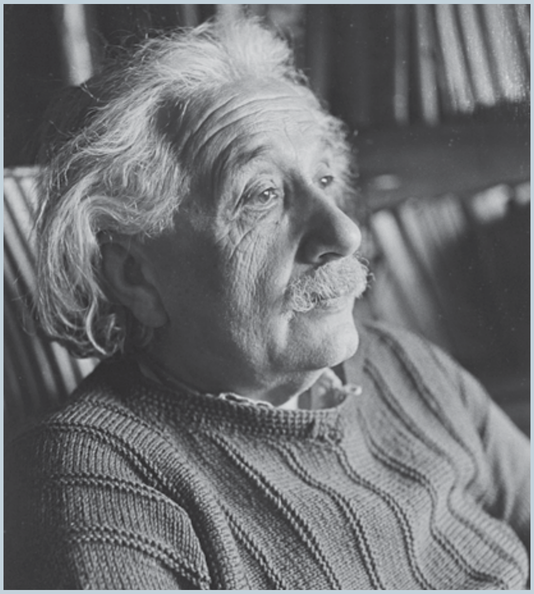
The year 2015 marks the 100th anniversary of the most profound of Einstein's intellectual feats, general relativity, a theory that explains the relationship between gravity and matter. With this work, Einstein unleashed extraordinary new concepts such as black holes, the Big Bang, the bending of light by galaxies, and the rippling of gravitational waves through space, all consequences of the theory and the mathematical equations that describe it.

Einstein's theory explains how matter, in the form of galaxies, suns, planets and other objects, creates gravitational fields in the fabric of the universe and how these gravitational fields in turn control the behavior of matter. His ideas, set forth in a series of lectures in late 1915, were almost immediately applied to describe, for example, the unconventional orbit of the planet Mercury.

But although the theory is easy to explain in words, the underlying math, in the form of partial differential equations, is considerably more complicated. "Einstein's equations provide a tremendous number of deep problems for mathematicians," said Sergiu Klainerman, the Eugene Higgins Professor of Mathematics. "These are some of the most difficult equations there are, by far."

Klainerman is one of a group of Princeton mathematicians working on general relativity that includes Rodnianski; Mihalis Dafermos, the Thomas D. Jones Professor of Mathematical Physics; Alexandru Ionescu, professor of mathematics; Stefanos Aretakis, assistant professor of mathematics; and several postdoctoral researchers and graduate students.

PHOTO BY HERMANN LANDSHOFF/ MÜNCHNER STADTMUSEUM, SAMMLUNG FOTOGRAFIE, ARCHIV LANDSHOF



Finding new ways to understand these equations could aid in the understanding of black holes, regions of extremely dense gravity that were predicted by Einstein's equations long before they were observed in the universe. One of the outstanding problems in general relativity is to explain mathematically how a process called gravitational collapse results in a black hole. "We want to understand what it is about the present that tells us a black hole will form in the future," Rodnianski said.

Another area of research is cosmic censorship, first posed by physicist Roger Penrose, which can be roughly translated as, "Whatever happens in a black hole stays in a black hole." In other words, mathematicians want to find proof that the intense gravity in a black hole cannot come out to wreak havoc in the universe. "It is a very comforting thought," Klainerman said, "but it is a very difficult mathematical problem, and one of our long-term objectives is to prove it or disprove it."

Dinner party **mathematics**

Imagine you are planning a banquet. You want to make sure that if two people don't like each other, they don't sit at the same table. To help with planning, you might draw a diagram with dots for guests and lines joining the pairs that are best kept apart.

Maria Chudnovsky studies mathematical objects called graphs, which consist of dots and lines, with each line connecting two dots. "A graph is a good tool to model real-life situations where the information comes in pairs," Chudnovsky said, "such as pairs of cities connected by a direct flight, pairs of people who know each other, or pairs of computers connected by an optical fiber."

Assigning the dots on the graph (or people at your banquet) into groups, so that no conflicts occur is, in mathematical parlance, called "coloring" the graph. That means coloring the dots so that no two dots, or nodes, of the same color are connected. (Think color-coding the tables, and then "coloring" the guests according to their assigned table.)

No efficient algorithm exists for coloring that applies to all situations, Chudnovsky said, but there are classes of graphs, such as one called the "perfect graphs," that behave particularly well with respect to coloring. While a graduate student at Princeton, Chudnovsky was part of a team — made up of Chudnovsky's Ph.D. adviser and Professor of Mathematics and Applied and Computational Mathematics Paul Seymour, Neil Robertson of Ohio State University, and Robin Thomas of the Georgia Institute of Technology — that solved a 40-year-old problem: a conjecture stating that there is always a simple reason why a graph is not perfect. The conjecture they proved is called "The Strong Perfect Graph Theorem."

"It was very exciting to accomplish this as a graduate student — it was as huge a shock as you can imagine," Chudnovsky said. She went on to become a professor at Columbia University before returning last year to Princeton, where she is now a professor of mathematics and the Program in Applied and Computational Mathematics. She receives funding for her research from the National Science Foundation and the United States-Israel Binational Science Foundation.

PHOTO BY DENISE APPLEWHITE

More recently, Chudnovsky has been exploring another question about perfect graphs. Can you efficiently color the dots with the smallest possible number of colors? The answer to this question is yes, but all the known algorithms use a complex technique known as combinatorial optimization. Might there be another technique that relies solely on graph theory? Recently, Chudnovsky and Columbia graduate student Irene Lo, with Frédéric Maffray at the Grenoble Institute of Technology, Nicolas Trotignon of École Normale Supérieure de Lyon, and Kristina Vušković of the University of Leeds, were able to make progress on this, designing such an algorithm for a large subclass of perfect graphs.

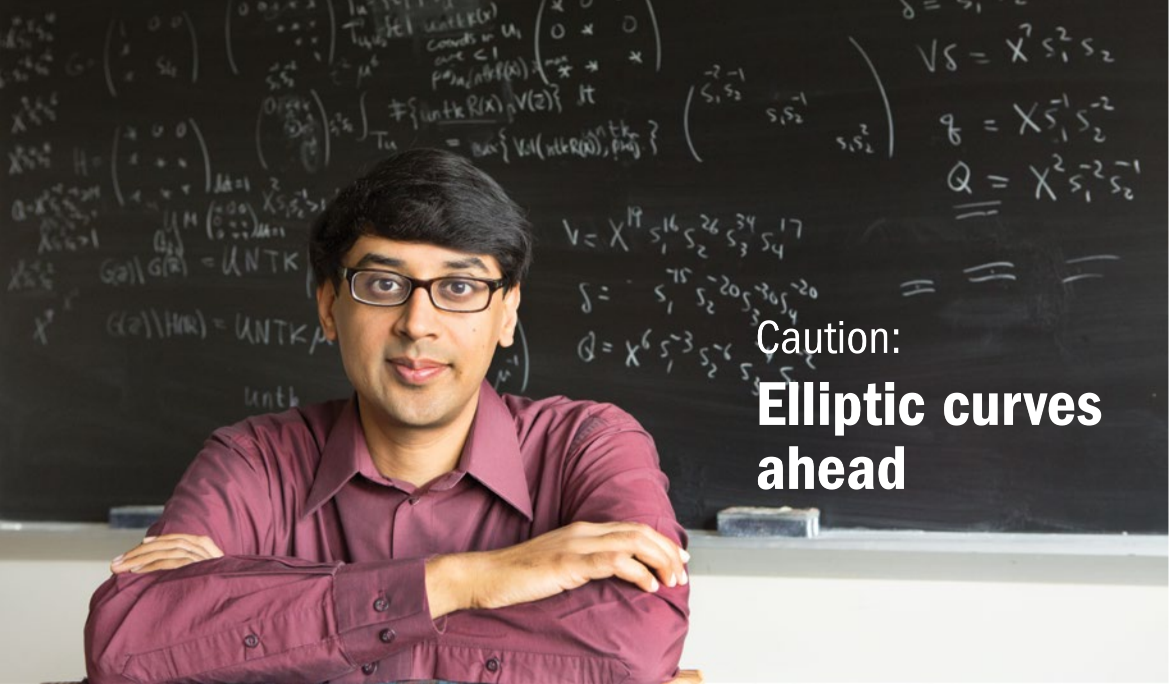
Advanced course: A sampling of mathematics research areas

Number theory: Strange patterns pervade our counting numbers. Why do prime numbers usually appear in pairs separated by one number (3 and 5, 5 and 7, 11 and 13, and so on)? Finding the hidden behaviors and patterns of numbers is the basis of number theory.

Analysis: Any signal, whether it is coming from a violin or from the stars, consists of multiple parts. For example, the sound from a violin consists of the tone plus the harmonics. Analysis enables the evaluation of signals in terms of their basic constituents.

Topology: We can think of shapes in terms of how many holes they have. (A donut has one hole, a figure 8 has two.) Topology describes shapes in terms of whether they retain their identity when they are stretched and deformed, but not ripped or cut.

The department is also strong in many other areas of mathematical research, including applied mathematics, dynamical systems, discrete mathematics, geometry and mathematical physics.



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Caution: **Elliptic curves ahead**

Manjul Bhargava was warned long ago never to think about math while driving. “I find doing mathematical research requires very deep concentration,” said Bhargava, the Brandon Fradd, Class of 1983, Professor of Mathematics. “It is almost like a meditative state.”

On a sunny afternoon in his office, however, Bhargava is lively and enthusiastic as he talks about his research. He is known about campus for his mathematical accomplishments (in 2014 he won the Fields Medal, considered the highest honor in mathematics) and for his popular freshman seminar, “The Mathematics of Magic Tricks and Games.”

But it is his research on curves — or more precisely, elliptic curves — that has the mathematical world taking notice. Beautiful and practical, elliptic curves are at the forefront of mathematics research and are increasingly being used in cryptography algorithms to secure the privacy of online transactions.

They are also worth $1 million in prize money to the person who can explain them. Elliptic curves are at the heart of one of the seven greatest unsolved mathematical puzzles of our time, according to the Clay Mathematics Institute, which offers the prize. A team including Bhargava and Princeton Professor of Mathematics Christopher Skinner, along with mathematician Wei Zhang of Columbia University and Arul Shankar, who earned his Ph.D. at Princeton and is now at Harvard University, is making exciting progress.

Elliptic curves have captivated mathematicians because they have a special property: If you draw a straight line through any two points on the curve, that line will always intersect the curve at exactly one more point. Not only is this unusual — no other kind of equation has this property — but cryptography schemes can use this property to encode passwords.

“There is no known set of instructions for how to solve these equations, or even whether or not it is possible to solve them,” said Skinner. Skinner and Bhargava receive support for their research from the National Science Foundation and the Simons Foundation.

So far, mathematicians have shown that some elliptic-curve equations have an infinite number of solutions that are “rational,” meaning they are either whole numbers or they are fractions. Other equations have a finite number of rational solutions. The $1 million prize will go to the person or team that confirms that there is a way to tell whether there are an infinite or finite number of rational solutions, thereby proving a mathematical idea known as the Birch and Swinnerton-Dyer conjecture.

So far, contributions from Bhargava, Skinner, Shankar and Zhang have found that the conjecture is true for 66 percent of elliptic curves. This is still a long way from proving that the conjecture is true for all curves, warned Bhargava. “We need some additional ideas before we can prove that the conjecture is true for all curves.” D

In memory of **John Nash**

A legendary fixture of Princeton’s Department of Mathematics, John Nash Jr. was renowned for his breakthrough work in mathematics and game theory as well as for his struggle with mental illness. In May 2015, Nash and his wife, Alicia, died in an automobile accident while returning from Oslo, Norway, where Nash had received the Abel Prize, one of the most prestigious honors in mathematics, from the Norwegian Academy of Science and Letters. “He had a profound originality as if he somehow had insights into developing problems that no one had even thought about,” said department chair David Gabai. “Beyond great originality, he demonstrated tremendous tenacity, courage and fearlessness.